# Rational and Semi-Rational Explanations of the Conjunction Fallacy: A Polycausal Approach

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**Abstract**

Conjunction fallacies (CF) have not only been a major obstacle in justifying the rationality of a Bayesian theory of belief update; they have also inspired a variety of theories on probability judgment and logical predication. Here we provide an overview of Bayesian logic (BL) as rational formulation of a pattern-based class of conjunction fallacies. BL is described here as a generalization of Bayesian Occam's razor. BL captures the idea that probabilities are sometimes used not extensionally but intensionally, determining the probabilistic adequacy of ideal logical patterns. It is emphasized that BL is a class of models that depend on representations and the meanings of logical connectives. We discuss open questions and limits of BL. We also briefly discuss whether other theories of the CF may be good supplementary theories of CFs (and predication) as well, if linked to functional explanations.

**Keywords:** probability judgments; biases; conjunction fallacy; inclusion fallacy; inductive logics; intensional logics; Bayesian logics; predication; strong sampling; categories; Lockean Thesis; rationality debate; Bayesian Occam's razor

## Extensional vs. Intensional Probabilities

### Extension vs. Intension

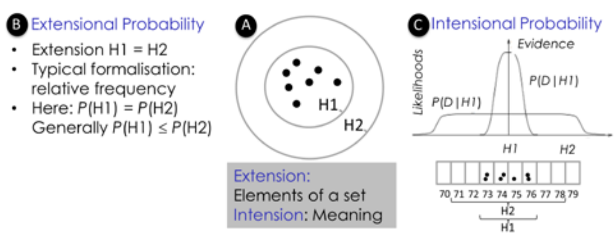


Figure 1: (A) Extensions as elements and intensions demarcated by set boundaries. (B) Characterization of extensional and (C) intensional probabilities (cf. McKay, 2003, 28).

Although less known to the psychologist than to the philosopher or logician, the notions of extension and intension (≠ "intention") have a tradition going back to Leibniz, Carnap, and Stegmüller; with several analogous terms proposed by others, such as 'meaning' and 'denotation' (Russell) or 'Sinn' and 'Bedeutung' (Frege). Extension refers to the elements of a set, and intension to the meaning, which may be symbolized by the area determined by the set boundaries (Figure 1A). Correspondingly, a set can be described extensionally by specifying its elements or intensionally by specifying one or several defining features.

### Extensional narrow norms of predication

Extensional approaches have long dominated set theory (Zermelo–Fraenkel set theory), logic (propositional logic), and probability theory (Kolmogorov's axiomatization). According to these extensional approaches, two sets are identical if they have the same elements (the same extension). Such an approach entails that any logically stronger (more specific) proposition implies any more general proposition (e.g., $A \wedge B \Rightarrow A \vee B$); and that any more specific hypothesis can never be more probable than a more general one.

Extensional logic and extensional probabilities have been proposed to provide universal criteria of rational predication. Predication attributes a predicate or logical combination of predicates to a subject. First, valid (assertive) predication of general logical relationships has traditionally often been linked to a logical truth-table definition of connectives (Frege, Russell, Whitehead, and Wittgenstein) (Table 1). Accordingly, if a conjunction $A \wedge B$ is true in a universe of discourse ($X$), no cases in $X$ fall outside of the corresponding set (the intersection) and the truth of the conjunction implies, for instance, the truth of the affirmation $A$ as well as of the disjunction $A \vee B$ (Table 1).

Table 1: Truth tables of some dyadic logical connectives: conjunctions, exclusive disjunctions, affirmations, and inclusive disjunctions

| *A* | *B* | *A* ∧ *B* | *A* >-< *B* | *A* | *A* ∨ *B* |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | F | F | T | T | T |
| F | T | F | T | F | T |
| F | F | F | F | F | F |

Assertive, contingent sentences, such as "Members of Species *X* are aggressive (*A*) AND curious (*B*)" (von Sydow & Fiedler, 2012) (in predicate logic: $\forall x\ A(x) \wedge B(x)$), can thus be falsified by a single observation (Popper, 1934; but see Oaksford & Chater, 2007). One problem in using (extensional) logic as adequacy criterion of contingent general predications is that they often involve exceptions (von Sydow, 2013). The problem of exception refers to the phenomenon that we seem to employ general sentences, like ravens are black, even if there are known exceptions, such as albino ravens. One solution to this problem is to replace the logical criterion of adequate predication by a high-probability criterion ($P(\text{Assertion}) > \varphi > .5$) (Schurz, 2001; cf. Adams, 1986; Oaksford & Chater, 2007, Pfeiffer, 2013; BL makes use of this idea, but for *intensional* probabilities).

Second, Kahneman & Tversky (1983) argued that any deviation from the probabilistic extensional conjunction rule, $P(A \wedge B) \leq P(A)$, involves a conjunction 'fallacy', even in the context of predication. It is argued that extensional logic and extensional probability are narrow norms (cf. Gigerenzer, 1996; Fiedler & von Sydow, 2013) if applied as adequacy criterion for rational predication (von Sydow, 2011, 2016). One problem of a direct application of an extensional probability criterion to predication is sample size. That is, (extensional) relative frequencies do not distinguish between 1/1 and 1000/1000 confirmative ravens.

However, the main problem with extensional probability is that predicates referring to subsets can never yield a higher probability than those referring to supersets. However, it should be possible do deem more specific hypotheses to be more adequate; otherwise one could never prefer a more specific hypothesis. And it seems absurd, if, for instance, "*X* are aggressive AND curious" ($A \wedge B$) could never be more probable and hence more adequate than "*X* are aggressive OR curious or both" ($A \vee B$). Likewise, the tautology (or 'verum') "*X* are aggressive or not, and they are curious or not" (*A*T*B*) by definition (with a maximal extensional probability of 1) could never be less adequate than a more suitable, specific hypothesis, independent of empirical evidence. Therefore a universal extensional high-probability criterion fails the requirement to be empirically informative.

## Intensional Probabilities of Bayesian Logic

### Bayesian Occam's Razor

The basic idea of a Bayesian Occam's razor already offers a partial solution to the problem of inclusion (Jeffreys & Berger, 1992; Tenenbaum & Griffiths, 2001; McKay, 2003; cf. Navarro et al., 2012). If a consequential region of hypotheses H1 is a subset of a consequential region of a hypothesis H2 (Figure 1C), even a subjective Bayesian account may be extensional in the sense of requiring that the more specific hypothesis can never be more probable than a more general one (this is sometimes called 'weak sampling'). However, if one treats the nested hypotheses nonetheless as *alternative* explanatory patterns (hypotheses) whose consequence regions may each have produced the data, the size of the consequential regions matters (sometimes called 'strong sampling'). In this case, data coherent with the specific (and also the general) hypothesis (cf. Figure 1C) is more likely to occur based on the more specific hypothesis: $P(\text{data}|\text{H1}) > P(\text{data}|\text{H2})$. If one additionally assigns an equal prior to these alternative hypotheses, $P(\text{H1}) = P(\text{H2})$, one can indeed get a higher posterior for the more specific hypothesis, $P(\text{H1}) > P(\text{H2})$. For extensional probabilities, by contrast, a more specific hypothesis could never obtain a higher probability than a more general one (inclusion rule). We even treat this rule as the defining feature of extensional probabilities. Since Bayesian Occam's razor (strong sampling) violates this rule, we may thus call it an 'intensional' probability. For intensional probabilities, actually both the size of the extension and that of the intension matter.

However, such a basic application of Bayesian Occam's razor does not allow for exceptions. A hypothesis is still falsified by a single disconfirmatory instance. But predications about contingent facts often allow for exceptions (von Sydow, 2013b). Otherwise one still could never prefer a more specific predication over a tautology. This problem of applying basic Bayesian Occam's razor (without noise) to real predications presumably explains why the predominant view holds that Bayesian accounts cannot rationally account for CFs (Fisk, 1996; Gigerenzer, 1998; Neace et al., 2008).

### BL as Generalized Bayesian Occam's Razor

Bayesian logic (BL, von Sydow, 2011, 2016; cf. von Sydow, 2013; von Sydow & Fiedler, 2012) addresses the problem of exception together with the problem of inclusion. Thus BL can be understood as using the 'natural' implications of Bayesian Occam's Razor together with the assumption that people are not interested in deterministic logical hypotheses but rather in noisy-logical hypotheses that are still similar to the deterministic hypotheses.

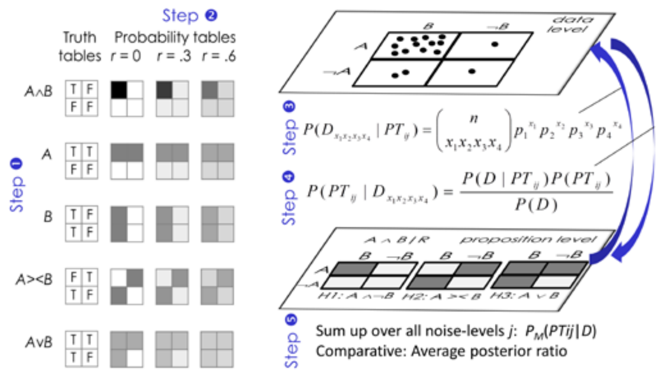

Figure 2: BL model sketch in 5 steps (cf. main text)

Step 1 of the model (Figure 2) turns deterministic truth tables into ideal (and still in some sense deterministic) probability tables (PTs). In the PTs above dark shadings represent a high probability (up to 1) and light shadings a low probability (down to 0). Step 1 only constructs PTs with noise level $r = 0$. Here, cells of a PT that correspond to logically false truth values (*F*) are assigned the value 0. Logically confirmatory cells (*T*) in these ideal representations, however, are assumed to be equi-probable (Johnson-Laird, Legrenzi, Girotto, Legrenzi, & Caverni, 1999). Step 1 (combined with Steps 3 and 4) already provides a basic Occam's razor solution. However, here, connectives are still falsified by single disconfirmatory events.

Step 2 addresses the problem of exceptions and constructs ideal noisy PTs (with $r > 0$) by adding noise to each cell of the ideal patterns from Step 1 and then renormalizing the PTs. There may be alternatives for modelling noise/acceptance levels (e.g., von Sydow, 2014), but as long as they yield similar results and address the problems of inclusion and exception simultaneously this seems a predominantly technical issue, and they should be seen as variants of the same computation model class of BL. Note that the PTs here are still ideal explanatory hypotheses composed of four cell probabilities adding up to 1 and a (second order) probability representing the belief in this PT-hypothesis (also adding up to 1, but now over all PT-hypotheses). We normally assume a flat prior distribution over all PTs (implying a flat prior distribution for both connectives and noise levels).

Step 3. The likelihood of each explanatory 2×2 PT (for $i \times j$ PTs, with $i$ modelled connectives and $j$ modelled equidistant noise levels) given the data, $P$(PT |$D$), can be calculated by a multinomial distribution. The data are i.i.d. observations in a 2×2 contingency matrix ($A$ vs. non-$A$, $B$ vs. non-$B$).

Step 4 uses Bayes' theorem to derive the posterior probabilities $P_I(PT|D)$ from the likelihoods $P(D|PT)$ and priors.

Step 5 sums up the posterior probabilities of all PTs created based on a particular logical hypothesis (over all $j$ noise levels). This results in (intensional and noise-tolerant) posterior probabilities for each of the $i$ connectives.

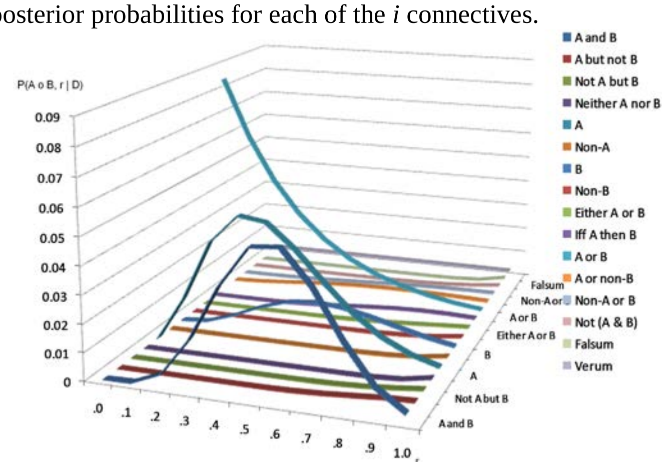


Figure 3. Intensional probabilities of logical hypotheses for 11 noise levels for a given 2× 2 data contingency matrix with the cells a, b, c, d, here [7, 3, 2, 0], and a flat prior.

Figure 3 shows resulting intensional probabilities for all modeled connectives at different noise levels (Step 4). If one assumes no additional weighting for particular noise-levels (Step 5) the marginal probabilities provide the intensional posteriors for the connectives. In the example, $P_i(A \wedge B)$, $P_i(A)$, and $P_i(A \vee B)$ have the highest overall probability. The intensional probabilities entail $P_i(A \wedge B) > P_i(B)$, although the extensional conjunction rule requires $P_E(A \wedge B) > P_E(B)$. '$A$ or $B$ (or both)' ($A \vee B$) is (intensionally) most probable at low levels of noise, but '$A$ and $B$' is most likely at higher levels of noise. Thus noise priors – for instance a belief in deterministic relationships – may change the intensional probability assessment.

## Findings Corroborating Standard Dyadic BL

Bayesian Logic, in its outlined main version, is an inductive logic providing intensional probabilities of dyadic logical connectives. The connectives relate two dichotomous events, and the model input are priors or frequencies (or equivalent Dirichlet-distributed, degrees of belief). BL provides a rational reconstruction of a class of pattern-based conjunction fallacies in line with Bayesian updating (cf. Hartmann, & Meijs, 2012, for another rational Bayesian model of CFs, based on source reliability). BL in a way detects the noisy-logical pattern that is most 'similar' to the data (actually $P(PT|D)$).

Some comments may be appropriate: First, if participants must rank the probability of two nested logical connectives (e.g., a conjunction and one of its conjuncts; Tversky & Kahneman, 1983), using intensional probabilities, $P_I$, instead of extensional ones, $P_E$, is not fallacious; both are probabilities. We here continue to speak of conjunction 'fallacies' only for reasons of convenience. If the previous argument is correct, it is even reasonable, in the context of predication and looking for the most adequate connective, to apply *intensional* probability, since it serves a reasonable function (providing an empirically informative probabilistic adequacy criterion for predication). Second, the intensional probabilities only supplement extensional probabilities. BL is likewise based on the standard extensional axioms of probability (Kolmogorov's axioms), but applies these axioms not on the level of extensions, but on that of probabilities of alternative logical hypotheses. (Similarly, in Bayes nets one may apply hypotheses probabilities to graphs without invalidating the underlying joint probability matrix.[1]) Third, BL is formulated not only as a normative but also as a descriptive (computational-level) theory of probability judgments concerning logical predications. However, the claim is, of course, not that people have a deliberate analytic understanding of BL. Their judgments may be roughly reasonable, as our perception system makes reasonable inferences without requiring conscious calculations. Moreover, people may merely be using something similar to intensional probabilities; and it needs to be explored whether they perhaps use some roughly related heuristic that only approximates BL.

One major finding that seems unique to pattern-based CFs advocated by BL, was BL's various predictions for conjunction fallacies. For instance, von Sydow (2011) showed that CFs occurred even with clearly defined subsets, clear logical hypothesis formulations, and data transparently presented in a contingency table (cf. Sloman et al., 2003). Moreover, the results confirmed predicted conditions of a dominant occurrence of a high proportion of double CFs, sample-size effects, and the results for negated propositions.

A second group of corroborations of BL concerns the generalization of (pattern-based) conjunction fallacies to other logical inclusion fallacies (von Sydow, 2009, 2013b, 2016; von Sydow & Fiedler, 2012). Figure 4 illustrates that logically there are many more logical inclusion relations than those involved in CFs, and thus many possible inclusion 'fallacies'. For instance, von Sydow (2016) corroborated that there is a more general system of inclusion fallacies broadly in line with BL, and that this system could not be explained by other major theories. Von Sydow & Fiedler (2012) applied this idea to sequential learning and repeated judgments, and von Sydow (2013b) has shown the

[1] Actually an analogous model of subjective belief update of the cell probabilities (using Dirichlet distributions) which is then compared to ideal patterns yields the same results (not elaborated here).

applicability of a pattern account even if numbers were not provided explicitly.

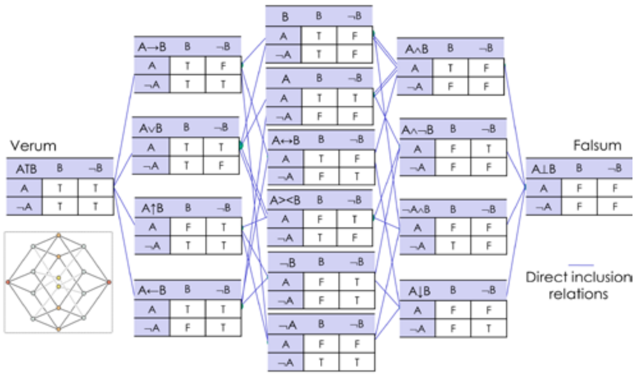


Figure 4. Inclusion relations between all 16 combinatorically possible dyadic logical connectives.

## BL as Model Class and Future Avenues of Research

### Monadic Dichotomous BL and Conjunctions as Combination of Marginals

This section supports the view that polysemous meanings of ordinary-language connectives play an important role in the CF debate, that this is compatible with BL, and that intensional BL even offers additional differentiations.

The intensional idea of BL (or the 'pattern idea') can also be applied to the simpler representation of monadic logic (thereby linking to the literature on generics). Whereas standard dyadic logic concerns all possible bivariate, two-valued (T, F) patterns in a $2 \times 2$ matrix (cf. Fig. 4), monadic logic concerns a single event only (a $2 \times 1$ matrix). The formalized intensional monadic BL (von Sydow, 2014, cf Tessler, & Goodman, 2016) predicts inclusion 'fallacies': e.g., our example (Figure 5, Panel A), $P_I(A) > P_I(A$ Tautology non-$A)$, or formulated as $P$(People in this group are artists) > $P$(… are artists or non-artists) (von Sydow, 2015).

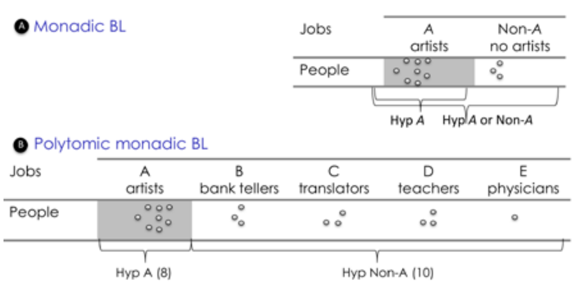


Figure 5. Illustration of representations modelled in (dichotomous) monadic BL and (polytomous) monadic BL

The model starts with a Beta prior, a binomial likelihood, and therefore continues with a Beta posterior (Figure 6). We pursued a slightly different approach of modelling noise levels here (without changing the pattern idea). We used integrals of the same size over parts of the posterior belief-distribution extending from 0, .5, or 1, in order to formalize the alternative hypotheses '$A$', '$A$ or non-$A$', and 'non-$A$' (Figure 6, red marks).

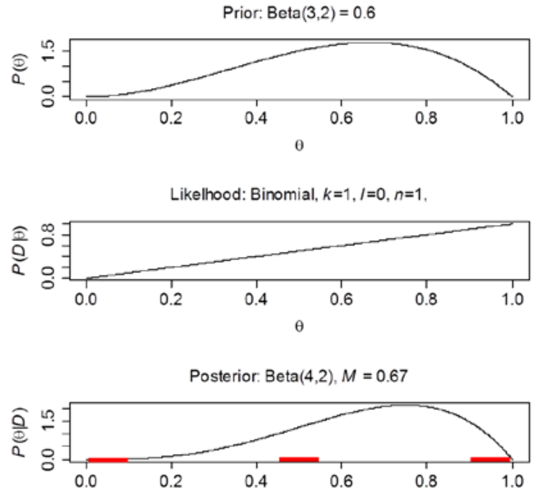


Figure 6. Monadic BL and example for prior, likelihood, posterior, and the red integrals.

Based on monadic BL (Figure 5A, 6) a new meaning of the AND-connective based on two marginal probabilities has been proposed (Figure 7B, von Sydow, 2014).


Figure 7. Dyadic and marginal meaning of 'AND'.

This meaning is in line with the approach that the ordinary language "and" is logically polysemous and may refer to the dyadic conjunction, the disjunction or the sum (Hertwig et al., 2008; von Sydow, 2015). The conjunction of monadic affirmations (von Sydow, 2014a) actually refers to the same cells as the inclusive-disjunction interpretation proposed earlier, whereas this proposal intensionally makes different predictions. Note that it seems possible to link the dyadic and marginal interpretation (Figure 7) to different formulations favouring a more dyadic ("the pub is visited by people who are young and (also) male") or a more marginal interpretation ("the pub is visited by young people and is visited by male people"). Additionally, we supported already known usages of AND as an addition of classes in an intensional polytomous context (von Sydow, 2015).

### Polytomous Monadic BL

Figure 5 (Panel B) points to a further interesting perspective suggested by an approach that looks at not only (relative) cardinality of extensions (e.g., relative frequencies) but also the size of the (represented) intension. For the data shown in Panel B, both extensional and here also intensional (dichotomous) monadic BL would not allow to predict $P$(… are $A$) > $P$(are Non-$A$). However, polytomous monadic BL, which here assumes polytomous representations for the negation, predicts even this. Moreover, it predicts, for instance, P(… are $A$) > $P$(are $A \vee D \vee E$). Von Sydow (2015) elaborated a model in other regards analogous to standard BL, tested many patterns, and contrasted the predictions of BL with a confirmation account (Tentori et al., 2013). The results clearly corroborated BL (in some examples even independently of the various measures of confirmation). More generally, this account emphasizes that BL assigns high intensional probabilities to patterns most adequately describing a situation, in the sense of having a *relatively* high probability while taking intension-size into account.

## Some Further Open Questions

(1) Variants of BL. The relation of two kinds of variants of BL needs further elaboration and scrutiny. First, BL has been shown to be an intensional model class for logical predication that depends on dyadic versus monadic, and dichotomous versus polytomous representations. Second, we actually used different model variants to model noise (cf. von Sydow, 2007 (cf. 2011), 2014, 2016; and there may be further variants). Despite impressive fits of all models (and, as far as I can see, only minor differences between them), one may design experiments to differentiate between these second kinds of model variants as well.

(2) BL and Conditionals: One may apply the idea of BL to further representations. For instance, von Sydow (2014) proposed an intensional model of conditionals, building on additional representational assumptions about conditionals. Inspired by mental model theory, it was suggested to differentiate between basic conditionals based on conditional probability alone, and full models based on Delta p (or causal Power). In an intensional setting, when the probability of conditionals is compared with the probability of other logical connectives, the intensional version of this model requires testing. The Bayesian logic of conditionals may also throw light on the paradoxes of implication (cf. von Sydow, 2009).

(3) BL and Reasoning: Here BL is presented as an inductive logic only, not directly applicable to reasoning without further assumptions. However, only a few assumptions may be needed to make BL fruitful in this field as well. Extensional or intensional premises may simply change the joint probability matrix (or frequency matrix), and the update may be based on standard conditionalization, Jeffry conditionalisation, or Kullback-Leibler distance-reduction. Based on resultant joint probability distribution (or the equivalent frequency distribution), one may infer intensional probabilities for resulting connectives using BL. The inferences would be based on prior beliefs and on the logical form of the added premise (cf. dual process theories; e.g., Singermann, Klauer, & Beller., 2016). The variety of advocated representations (extensional, intensional; dyadic, monadic; dichotomous and polytomous, etc.) and alternations between these modes needs future attention. For instance, one may intensionally believe in the dyadic hypothesis "(normally) $A$ and $B$". In line with Foley (1992; cf. the Lockean thesis) this does not need to imply a high probability of the composing single (dyadic) hypotheses $A$ ("$A$ and $B$ or $A$ and non-$B$"). In contrast, the *monadic* hypothesis $A$, in such situations, would always have a high probability as well (cf. von Sydow, 2014).

These suggestions demand further elaboration, but the prior confirmation of the BL and its variants suggests that they may be helpful in this domain as well.

## Other Theories of CFs: Polycausal Semi-Rational Suggestions

It has been shown repeatedly that the results confirming BL could not be explained by other major theories of the CF (von Sydow, 2011, 2015, 2016). This does not entail that these theories do not have a reasonable domain of application. There may well be several causes of CFs or, more generally, of inclusion fallacies (IFs; cf. von Sydow, 2016).

BL itself is 'polycausal' in the sense that the modelling depends on the representation. One should use different formalizations for different scales and sampling assumptions (von Sydow, 2015, cf. Tessler & Nelson, 2016). In particular, representation of classes matters due to intensionality, with different results for dichotomous and polytomous events. Moreover, BL is consistent with various interpretations, for instance, of the ordinary conjunctions (Hertwig et al., 2008; von Sydow, 2015) and even adds new candidates to this list (von Sydow, 2014).

There are further theories of CFs claiming that the target measure $P(H|D)$ is substituted by other measures, such as inverse probability, confirmation, or averaging (cf. also Costello & Watts, 2014). The predictions of these theories are thus more difficult to defend from a rational point of view. BL's substitution of $P(H|D)$ by intensional $P_I(H|D)$ rather than by extensional $P_E(H|D)$ involves no substitution at all, only a specific interpretation, and, as outlined before, a reasonable one. In contrast, replacing $P(H|D)$ by $P(D|H)$, as suggested by an inverse-probability account, or by, for instance, $P(H|D)$-$P(H|D)$, as in a confirmation account (Lagnado & Shanks, 2003; Tentori et al., 2013), seems less rational, since this involves an illicit replacement. Nonetheless, if this replacement would be linked to a functional explanation why and when this replacement should occur, this may be seen as semi-rational as well. An interest in adequately describing a logical relationship between given features, provides a BL context. An interest in a particularly high probability of a feature given a class may provide a context for inverse probabilities (or an inverse pattern account). And an interest in the surprisingness of a feature may be a context for a confirmation account (or a pattern-confirmation account). Although functional explanations and application conditions are currently still often missing in these theory presentations, they may be reformulated as semi-rational accounts of the

CF as well. These theories may exist in unproblematic cohabitation with the even more rational account of BL and, perhaps, like BL, with an extensional usage of probabilities.

## Summary

Whereas previous presentations of BL were mainly concerned with presenting specific empirical findings, the present account tries to provide more of an overview. BL is presented here in an overview as an intensional account that generalizes Bayesian Occam's razor in the field of logical predications. BL is posited not as a specific model but rather as a model class sensitive to representation, open to further extensions, and predicting many still unexplored effects. Furthermore, it was emphasized that BL is in line with theories assuming various meanings of connectives while fostering new proposals and opening up many new avenues of research. Finally, it was argued that the disconfirmation of other theories when testing BL does not at all rule out the adequacy of other accounts of CFs. It was suggested that some other accounts, if they would more clearly specify functional explanations, may count as semi-rational theories of CFs. Theories that pretend to provide a single algorithmic account of CFs underestimate the contextuality and goal dependence of such judgments. In the future, a polycausal theory of CFs needs to be elaborated, including rational accounts, involving BL in its various versions (relating to different representations), the mentioned semi-rational accounts (as well as a noise + probability account), and, perhaps, completely irrational accounts.

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## References

Adams, E. W. (1986). On the logic of high probability. *Journal of Philosophical Logic, 15*, 255-279.

Costello, F. and Watts, P. (2014). Surprisingly rational: Probability theory plus noise explains biases in judgment. *Psychological Review, 121*(3):463–480.

Fiedler, K. & von Sydow, M. (2015). Heuristics and Biases: Beyond Tversky and Kahneman's (1974) Judgment under Uncertainty (pp. 146-151). In: M. W. Eysenck & D. Groome. *Cognitive Psychology: Revisiting the Classical Studies*. Los Angeles, London: Sage.

Fisk, J. E. (1996). The conjunction effect: Fallacy or Bayesian inference? *Organizational Behavior and Human Decision Processes, 67*, 76–90.

Foley, R. (2009). Beliefs, Degrees of Belief, and the Lockean Thesis. In: F. Huber, C. Schmidt-Petri (eds.), *Degrees of Belief*, Synthese Library 342, Heidelberg: Springer.

Gigerenzer, G. (1996). On narrow norms and vague heuristics. *Psychological Review, 103*, 592-596.

Gigerenzer, G. (1998). Ecological intelligence: An adaptation for frequencies. In D. D. Cummins & C. Allen (Eds.), *The evolution of mind.* New York: Oxford University Press.

Hartmann, S., & Meijs, W. (2012). Walter the banker - the conjunction fallacy reconsidered. *Synthese, 184*, 73–87.

Hertwig, R., Benz, B., & Krauss, B. S. (2008). The conjunction fallacy and the many meanings of and. *Cognition, 108*, 740-753.

Johnson-Laird, P. N., Legrenzi, P., Girotto, V., Legrenzi, S. M., & Caverni, J.-P. (1999). Naive probability: A mental model theory of extensional reasoning. *Psychological Review, 106*, 62–88. DOI:10.1037/0033-295X.106.1.62

Lagnado, D. & Shanks, D. (2003). The influence of hierarchy on probability judgments. *Cognition, 89*, 157–178.

MacKay, D. J. C. (2003). Information Theory, Inference, and Learning Algorithms. Cambridge University Press.

Navarro, D., Dry, M. J., & Lee, M. (2012). Sampling Assumptions in Inductive Generalization. *Cognitive Science, 36(2)*, 187–223.

Neace, W. P., Michaud, S., Bolling, L., Deer, K., & Zecevic, L. (2008). Frequency formats, probability formats, or problem structure? A test of the nested-sets hypothesis in an extensional reasoning task. *Judgment and Decision Making, 3*, 140-152.

Oaksford, M., & Chater, N. (2007). *Bayesian rationality. The probabilistic approach to human reasoning*. Oxford: Oxford University Press.

Pfeifer, N. (2013). The new psychology of reasoning: A mental probability logical perspective. *Thinking & Reasoning, 19(3-4)*, 329-345.

Schurz, G. (2001). Normische Gesetzeshypothesen und die wissenschafts-philosophische Bedeutung des nichtmonotonen Schliessens. *Journal for General Philosophy of Science, 32*, 65-107.

Sloman, S. A., Over, D., Slovak, L., & Stibel, J. M. (2003). Frequency illusions. *Organizational Behavior and Human Processes, 91*, 296-309.

Singmann, H., Klauer, K. C., & Beller, S. (2016). Probabilistic Conditional Reasoning: Disentangling Form and Content with the Dual-Source Model. Cognitive Psychology, 88, 61-87. doi:10.1016/j.cogpsych.2016.06.005

Tenenbaum, J. & Griffiths, T. (2001a). Generalization, similarity, and Bayesian inference. *Behavioral and Brain Sciences, 24(4)*, 629-640.

Tentori, K., Crupi, V., & Russo, S. (2013). Determinants of the conjunction fallacy: Confirmation versus probability. *Journal of Experimental Psychology: General, 142*, 235-255.

Tessler, M. H. & Goodman, N. D. (2016). Communicating generalizations about events. *In Proceedings of the Thirty-eighth Annual Conference of the Cognitive Science Society* (1655–1660). Austin, TX: Cognitive Science Society.

Tversky, A., & Kahneman, D. (1983). Extensional versus intuitive reasoning: The conjunction fallacy in probability judgment. *Psychological Review, 90*, 293-315.

von Sydow, M. (2011). The Bayesian Logic of Frequency-Based Conjunction Fallacies. *Journal of Mathematical Psychology, 55(2)*, 119-139. doi:10.1016/j.jmp.2010.12.001

von Sydow, M. (2009). On a General Bayesian Pattern Logic of Frequency-Based Logical Inclusion Fallacies. In *Proceedings of the Thirty-First Annual Conference of the Cognitive Science Society* (pp. 248-253). Austin, TX: Cognitive Science Society.

von Sydow, M. (2013a). System implemented by a processor controlled machine for inductive determination of pattern probabilities of logical connectors. *United States Patent* (Issued), No. US 8,463,734 B2.

von Sydow, M. (2013b). Logical Patterns in Individual and General Predication. In M. Knauff, et al. (Eds.), *Proceedings of the Thirty-Fifth Annual Conference of the Cognitive Science Society* (pp. 3693-3698). Austin TX: Cognitive Science Society.

von *Sydow*, M. (2014a). Is there a Monadic as well as a Dyadic Bayesian Logic? Two Logics Explaining Conjunction 'Fallacies'. *Proceedings of the Thirty-Sixth Annual Conference of the Cognitive Science Society* (pp. 1712-1717). Austin TX: Cognitive Science Society.

von Sydow, M. (2014b). Bayesian Mental Models of Conditionals. *Cognitive Processing, 15* (Special Issue: Proceedings of the 12th Biannual Conference of the German cognitive science society), 148-151. doi:10.1007/s10339-014-0632-2.

von Sydow, M. (2015). Pattern Probabilities for Non-Dichotomous Events: A New Rational Contribution to the Conjunction Fallacy Debate. *Proceedings of the 37th Annual Conference of the Cognitive Science Society* (pp. 2511-2516). Austin, TX: Cognitive Science Society.

von Sydow, M. (2016). Towards a Pattern-Based Logic of Probability Judgements and Logical Inclusion "Fallacies". *Thinking & Reasoning, 22(3)*, 297-335. doi:10.1080/13546783.

von Sydow, M. & Fiedler, K. (2012). Bayesian Logic and Trial-by-trial Learning. *Proceedings of the 34th Annual Conference of the Cognitive Science Society* (1090 - 1095). Austin TX: Cognitive Science Society.